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JACOB'S LADDERS AND NEW ORTHOGONAL SYSTEMS GENERATED BY JACOBI POLYNOMIALS

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Abstract. Is is shown in this paper that there is a connection between the Riemann zeta-function $\zeta\left(\frac{1}{2}+it\right)$ and the classical Jacobi's polynomials, i.e. the Legendre polynomials, Chebyshev polynomials of the first and the second kind, ...

1. The result

1.1. In this paper we obtain some new properties of the signal

$$Z(t)=e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right)$$

that is generated by the Riemann zeta-function, where

$$\vartheta(t)=-\frac{t}{2}\ln\pi+\operatorname{Im}\ln\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right)=\frac{t}{2}\ln\frac{t}{2\pi}-\frac{t}{2}-\frac{\pi}{8}+\mathcal{O}\left(\frac{1}{t}\right).$$

Let us remind that

$$\tilde{Z}^2(t)=\frac{\mathrm{d}\varphi_1(t)}{\mathrm{d}t},\ \varphi_1(t)=\frac{1}{2}\varphi(t)$$

where

$$\tilde{Z}^2(t)=\frac{Z^2(t)}{2\Phi'_\varphi[\varphi(t)]}=\frac{\left|\zeta\left(\frac{1}{2}+it\right)\right|^2}{\left\{1+\mathcal{O}\left(\frac{\ln\ln t}{\ln t}\right)\right\}\ln t} \tag{1.1}$$

(see [1], (3.9); [3], (1.3); [7], (1.1), (3.1), (3.2)), and $\varphi(t)$ is the Jacob's ladder, i.e. a solution of the nonlinear integral equation (see [1])

$$\int_0^{\mu[X(T)]}Z^2(t)e^{-\frac{2}{X(T)}t}\mathrm{d}t=\int_0^T Z^2(t)\mathrm{d}t.$$

1.2. The system of the Jacobi polynomials

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right],$$
$$x\in[-1,1],\ n=0,1,2,\dots,\ \alpha,\beta>-1$$

is the well-known system of orthogonal polynomials on the segment $x\in[-1,1]$ with the weight function

$$(1-x)^\alpha(1+x)^\beta,$$

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i.e. the following formulae hold true (comp. [18])

(1.2)

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)\mathrm{d}x=0,\ m\not= n,$$

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta\left[P_n^{(\alpha,\beta)}(x)\right]^2\mathrm{d}x=\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}.$$

It is shown in this paper that the $\tilde{Z}^2$-transformation of the Jacobi's polynomials generates a new system of orthogonal functions connected with $|\zeta\left(\frac{1}{2}+it\right)|^2$. In this direction, the following theorem holds true.

Theorem. Let $x=t-T-1,\ t\in[T,T+2]$ and

$$\varphi_1\{[\mathring{T},\overset{\circ}{\widehat{T+2}}]\}=[T,T+2],\ T\geq T_0[\varphi_1].$$

Then the system of functions

$$P_n^{(\alpha,\beta)}(\varphi_1(t)-T-1),\ t\in[\mathring{T},\overset{\circ}{\widehat{T+2}}],\ n=0,1,2,\dots$$

is the orthogonal system on $[\mathring{T},\overset{\circ}{\widehat{T+2}}]$ with the weight function given by

$$(T+2-\varphi_1(t))^\alpha(\varphi_1(t)-T)^\beta\tilde{Z}^2(t),$$

i.e. the following system of new-type integrals

$$\begin{gathered}\int_{\mathring{T}}^{\overset{\circ}{\widehat{T+2}}}P_m^{(\alpha,\beta)}(\varphi_1(t)-T-1)P_n^{(\alpha,\beta)}(\varphi_1(t)-T-1)\\ (T+2-\varphi_1(t))^\alpha(\varphi_1(t)-T)^\beta\tilde{Z}^2(t)\mathrm{d}t=0,\ m\not= n,\end{gathered}$$

(1.3)

$$\begin{gathered}\int_{\mathring{T}}^{\overset{\circ}{\widehat{T+2}}}\left[P_n^{(\alpha,\beta)}(\varphi_1(t)-T-1)\right]^2(T+2-\varphi_1(t))^\alpha(\varphi_1(t)-T)^\beta\tilde{Z}^2(t)\mathrm{d}t=\\ =\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)},\\ m,n=0,1,2,\dots,\ \text{for all } \mathring{T}\geq\varphi_1^{-1}(T),\ T\geq T_0[\varphi_1]\end{gathered}$$

is obtained, where $\varphi_1(t)-T-1\in[-1,1]$, and

$$\rho\{[-1,1];[\mathring{T},\overset{\circ}{\widehat{T+2}}]\}\sim T,\ T\to\infty. \tag{1.4}$$

Remark 1. This Theorem gives the contact point between the functions $\zeta\left(\frac{1}{2}+it\right)$, $\varphi_1(t)$ and the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$.

1.3. The seconf formula in (1.3) via the mean-value theorem (comp. (1.1) leads to

Corollary 1.

(1.5)

$$\begin{aligned}&\int_{\mathring{T}}^{\overset{\circ}{\widehat{T+2}}}\left[P_n^{(\alpha,\beta)}(\varphi_1(t)-T-1)\right]^2(T+2-\varphi_1(t))^\alpha(\varphi_1(t)-T)^\beta\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t\sim\\ &\sim\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}\ln\mathring{T},\ \mathring{T}\to\infty\\ &n=0,1,2,\dots.\end{aligned}$$

This paper is a continuation of the series [1]-[17].

2. Orthogonal systems generated by Legendre polynomials

If $\alpha=\beta=0$ then $P_m^{(\alpha,\beta)}(x)=P_n(x)$ is the Legendre polynomial. In this case, our Theorem implies the following

Corollary 2. The system of functions

$$P_n(\varphi_1(t)-T-1),\ t\in[\mathring{T},\overset{\circ}{\widehat{T+2}}],\ n=0,1,2,\dots$$

is the orthogonal system on the segment $[\mathring{T},\overset{\circ}{\widehat{T+2}}]$ with the weight function $\tilde{Z}^2(t)$, i.e. the following system of new-type integrals

$$\begin{aligned}&\int_{\mathring{T}}^{\overset{\circ}{\widehat{T+2}}}P_m(\varphi_1(t)-T-1)P_n(\varphi_1(t)-T-1)\tilde{Z}^2(t)\mathrm{d}t=0,\ m\neq n,\\ &\int_{\mathring{T}}^{\overset{\circ}{\widehat{T+2}}}[P_n(\varphi_1(t)-T-1)]^2\tilde{Z}^2(t)\mathrm{d}t=\frac{2}{2n+1},\\ &m,n=0,1,2,\dots,\ \text{for all }\mathring{T}\geq\varphi_1^{-1}(T),\ T\geq T_0[\varphi]\end{aligned}\tag{2.1}$$

is obtained.

From the second formula in (2.1) we obtain (comp. (1.5))

Corollary 3.

$$\begin{aligned}&\int_{\mathring{T}}^{\overset{\circ}{\widehat{T+2}}}[P_n(\varphi_1(t)-T-1)]^2\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t\sim\frac{2}{2n+1}\ln\mathring{T},\ \mathring{T}\to\infty,\\ &n=0,1,2,\dots.\end{aligned}\tag{2.2}$$

3. Orthogonal systems generated by Chebyshev polynomials of the first and the second kind

3.1. If $\alpha=\beta=-\frac{1}{2}$ then

$$P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)=\frac{(2n-1)!!}{(2n)!!}T_n(x),\ n=0,1,2,\dots$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind. In this case we obtain from our Theorem

Corollary 4. The system of functions

$$T_n(\varphi_1(t)-T-1),\ t\in[\mathring{T},\overset{\circ}{\widehat{T+2}}],\ n=0,1,2,\dots$$

is the orthogonal system of functions with the weight function

$$\frac{\tilde{Z}^2(t)}{\sqrt{1-(\varphi_1(t)-T-1)^2}},$$

i.e. the following system of the new-type integrals

(3.1)

$$\int_{\mathring{T}}^{\overset{\circ}{\widehat{T+2}}} T_m(\varphi_1(t)-T-1)T_n(\varphi_1(t)-T-1)\frac{\tilde{Z}^2(t)}{\sqrt{1-(\varphi_1(t)-T-1)^2}}\mathrm{d}t=0,\ m\neq n,$$

$$\int_{\mathring{T}}^{\overset{\circ}{\widehat{T+2}}} [T_n(\varphi_1(t)-T-1)]^2\,\frac{\tilde{Z}^2(t)}{\sqrt{1-(\varphi_1(t)-T-1)^2}}\mathrm{d}t=\left\{\begin{array}{lcl}\frac{\pi}{2} & , & n\geq 1,\\ \pi & , & n=0,\end{array}\right.$$

$m,n=0,1,2,\dots$, for all $\mathring{T}\geq\varphi_1^{-1}(T),\ T\geq T_0[\varphi_1]$, is obtained.

From the second formula in (3.1) we obtain (comp. (1.5))

Corollary 5.

$$\begin{aligned}&\int_{\mathring{T}}^{\overset{\circ}{\widehat{T+2}}} [T_n(\varphi_1(t)-T-1)]^2\,\frac{\left|\zeta\left(\frac{1}{2}+it\right)\right|^2}{\sqrt{1-(\varphi_1(t)-T-1)^2}}\mathrm{d}t\sim \\ &\sim\left\{\begin{array}{lcl}\frac{\pi}{2}\ln\mathring{T} & , & n\geq 1,\\ \pi\ln\mathring{T} & , & n=0,\end{array}\right.\quad \mathring{T}\to\infty,\ n=0,1,2,\dots\ .\end{aligned}\tag{3.2}$$

Remark 2. From (3.2) (see the second formula; since $T_0(x)=1$) the canonical asymptotic formula

$$\int_{\mathring{T}}^{\overset{\circ}{\widehat{T+2}}} \frac{\left|\zeta\left(\frac{1}{2}+it\right)\right|^2}{\sqrt{1-(\varphi_1(t)-T-1)^2}}\mathrm{d}t\sim\pi\ln\mathring{T},\ \mathring{T}\to\infty\tag{3.3}$$

follows.

3.2. If $\alpha=\beta=\frac{1}{2}$ then

$$P_n^{(\frac{1}{2},\frac{1}{2})}(x)=2\frac{(2n-1)!!}{(2n+2)!!}U_n(x),\ n=0,1,2,\dots$$

where $U_n(x)$ is the Chebyshev polynomial of the second kind. Then, from our Theorem, we obtain

Corollary 6. The system of the functions

$$U_n(\varphi_1(t)-T-1),\ t\in[\mathring{T},\overset{\circ}{\widehat{T+2}}],\ n=0,1,2,\dots$$

is the orthogonal system with the weight function

$$\sqrt{1-(\varphi_1(t)-T-1)^2}\tilde{Z}^2(t),$$

i.e. the following system of the new-type integrals

$$\begin{aligned} & \int_{\mathring{T}}^{\overset{\circ}{\widehat{T+2}}} U_m(\varphi_1(t)-T-1)U_n(\varphi_1(t)-T-1) \\ (3.4)\quad & \sqrt{1-(\varphi_1(t)-T-1)^2}\tilde{Z}^2(t)\mathrm{d}t=0,\ m\neq n, \\ & \int_{\mathring{T}}^{\overset{\circ}{\widehat{T+2}}} [U_n(\varphi_1(t)-T-1)]^2\sqrt{1-(\varphi_1(t)-T-1)^2}\tilde{Z}^2(t)\mathrm{d}t=\frac{\pi}{2}, \end{aligned}$$

$m,n=0,1,2,\dots$, for all $\mathring{T}\geq\varphi_1^{-1}(T),\ T\geq T_0[\varphi_1]$ is obtained.

From the second formula in (3.4) we obtain (comp. (1.5))

Corollary 7.

$$\begin{aligned} (3.5)\quad & \int_{\mathring{T}}^{\overset{\circ}{\widehat{T+2}}} [U_n(\varphi_1(t)-T-1)]^2\sqrt{1-(\varphi_1(t)-T-1)^2}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t\sim \\ & \sim\frac{\pi}{2}\ln\mathring{T},\ n=0,1,2,\dots,\ \mathring{T}\to\infty. \end{aligned}$$

Remark 3. From (3.5) (see the second formula; since $U_0(x)=1$) the canonical asymptotic formula

$$(3.6)\qquad \int_{\mathring{T}}^{\overset{\circ}{\widehat{T+2}}} \sqrt{1-(\varphi_1(t)-T-1)^2}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t\sim\frac{\pi}{2}\ln\mathring{T},\ \mathring{T}\to\infty$$

follows.

Remark 4. The Riemann zeta-function $\zeta\left(\frac{1}{2}+it\right)$ is connected with the classical orthogonal polynomials of Legendre and of Chebyshev by formulae (2.1), (2.2), (3.1)-(3.6), respectivelly.

4. Proof of Theorem

4.1. Let us remind that the following lemma holds true (see [6], (2.5); [7], (3.3)): for every integrable function (in the Lebeague sense) $f(x), x\in[\varphi_1(T),\varphi_1(T+U)]$ we have

$$(4.1)\qquad \int_T^{T+U} f[\varphi_1(t)]\tilde{Z}^2(t)\mathrm{d}t=\int_{\varphi_1(T)}^{\varphi_1(T+U)} f(x)\mathrm{d}x,\ U\in(0,T/\ln T]$$

where

$$(4.2)\qquad t-\varphi_1(t)\sim(1-c)\pi(t),$$

c is the Euler's constant and $\pi(t)$ is the prime-counting function. In the case (comp. Theorem) $T=\varphi_1(\mathring{T}), T+U=\varphi_1(\overset{\circ}{\widehat{T+U}})$, we obtain from (4.1)

$$(4.3)\qquad \int_{\mathring{T}}^{\overset{\circ}{\widehat{T+U}}} f[\varphi_1(t)]\tilde{Z}^2(t)\mathrm{d}t=\int_T^{T+U} f(x)\mathrm{d}x.$$

4.2. Putting

$$f(t)=P_m^{(\alpha,\beta)}(t-T-1)P_n^{(\alpha,\beta)}(t-T-1)(T+2-t)^\alpha(t-T)^\beta;\ U=2,$$

we have by (4.3) and (1.2) the following $\tilde{Z}^2$-transformation

$$\begin{aligned}
&\int_{\mathring{T}}^{\overset{\circ}{\widehat{T+U}}} P_m^{(\alpha,\beta)}(\varphi_1(t)-T-1)P_n^{(\alpha,\beta)}(\varphi_1(t)-T-1) \\
&(T+2-\varphi_1(t))^\alpha(\varphi_1(t)-T)^\beta\tilde{Z}^2(t)\mathrm{d}t= \\
&\int_T^{T+2} P_m^{(\alpha,\beta)}(t-T-1)P_n^{(\alpha,\beta)}(t-T-1)(T+2-t)^\alpha(t-T)^\beta\mathrm{d}t= \\
&\int_{-1}^1 P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\mathrm{d}x=0,\ m\not=n,
\end{aligned}$$

where $t=x+T+1$, i.e. the first formula in (1.3) holds true. Similarly we obtain the second formula in (1.3). Since (4.2) implies $\mathring{T}\to T$ then (1.4) holds true.

4.3. Next, from (4.2) we obtain

$$\begin{aligned}
&\mathring{T}-\varphi_1(\mathring{T})=\mathring{T}-T=\mathcal{O}\left(\frac{\mathring{T}}{\ln\mathring{T}}\right), \\
&\overset{\circ}{\widehat{T+2}}-\varphi_1(\overset{\circ}{\widehat{T+2}})=\overset{\circ}{\widehat{T+2}}-T-2=\mathcal{O}\left(\frac{\mathring{T}}{\ln\mathring{T}}\right),
\end{aligned}$$

and subsequently

$$\overset{\circ}{\widehat{T+2}}-\mathring{T}=\mathcal{O}\left(\frac{\mathring{T}}{\ln\mathring{T}}\right),$$

and for $\xi\in(\mathring{T},\overset{\circ}{\widehat{T+2}})$ we have

$$\ln\xi=\ln\mathring{T}+\mathcal{O}\left(\frac{\overset{\circ}{\widehat{T+2}}-\mathring{T}}{\mathring{T}}\right)=\ln\mathring{T}+\mathcal{O}\left(\frac{1}{\ln\mathring{T}}\right). \tag{4.4}$$

The property (4.4) was used in (1.5),(2.2),(3.2),(3.5).

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